THE COHOMOLOGY RING OF A FINITE GROUP[1,2]

BY

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1. **Introduction.** Golod [3] has proved that the cohomology ring of a finite p-group is finitely generated as a ring for coefficients in Z/p^nZ or in Z. Using the classifying space of the group, Venkov [9] has provided a short and elegant proof of the finite generation of $H^*(G, Z)$ for G in a class of groups including the class of finite groups. (Actually Venkov proves a theorem for coefficients in Z/pZ. However, if emphasis is placed on modules with ascending chain condition rather than on finitely generated rings, it is easy to modify his proof so as to avoid restriction of the coefficient domain.)

In this paper I present a proof of the finite generation of $H^*(G, Z)$ for G a finite group; this proof is independent of the above mentioned efforts and its main interest is that it is completely within the cohomology theory of groups proper—i.e., it is purely algebraic. In such a context it is natural to generalize the theorem as to coefficient domain so as to prove the "natural" theorem for the theory. I also derive certain consequences of this theorem which are of interest in themselves.

The proof is as follows. The problem is reduced from an arbitrary finite group to a p-group by the Sylow subgroup argument in cohomology suitably modified. For the case of a p-group it is possible to proceed inductively by use of the spectral sequence of a group extension. One may prove successively that each E_r, $r \geqq 2$, is a finitely generated ring (or, equivalently, a noetherian ring), but whether or not this remains true for the E_∞ term is not determined. The problem is reduced thereby to showing that the spectral sequence stops; i.e., $E_r = E_\infty$ for some $r < \infty$. In the special case that the normal subgroup is cyclic and in the center of the group this amounts to showing that restriction to that subgroup is surjective in some positive even dimension.

This basic lemma is proved by a group theoretical argument. The group is imbedded in a certain wreath product so that the central cyclic subgroup gets carried into the center. This reduces the problem to a certain computation for the cohomology of wreath products.

Some remarks about notation are necessary. The symbol $\sum$ will always refer to direct sums unless otherwise stipulated. $\otimes$ will refer to tensor product over the ring of integers. An asterisk (*) in place of an index will indicate

Received by the editors March 25, 1961.

(1) A modified version of this paper was presented to the Mathematics Department of Harvard University in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

(2) Major parts of the work in this paper were supported in part or in whole by the National Science Foundation.

summation with respect to that index. Finally $H^*(G, A)$ will refer to the so-called unmodified cohomology groups.

2. **Noetherian modules.** This section is a collection of generally known facts about graded and filtered rings and modules. They are restated in a form most convenient for the purposes of this paper. Also some conventions are established for simplifying the terminology.

Generally the notation of [2] will be employed. As usual all rings have units and all modules are unitary. We shall deal only with *left* modules as this is no essential restriction. Graded objects are assumed to have non-negative gradings; that is, no negative superscript appears in a nonzero term.

Let

$$R = \sum_{n \geq 0} R^n = F^0R \supseteq F^1R \supseteq \cdots \supseteq F^sR \supseteq \cdots$$

be a singly graded filtered ring; that is, the ring structure, grading, and filtration of R are consistent with one another in the usual way. Then we may form the doubly graded ring

$$E_0(R) = \sum_{i,j \geq 0} E_0^{i,j}(R) = \sum_{i,j \geq 0} F^iR^{i+j}/F^{i+1}R^{i+j}.$$

(The doubly graded group $E_0(R)$ defined in [2] becomes a doubly graded ring if the appropriate product is defined by appeal to representatives in the relevant F^iR^{i+j}.) Similarly if A is a singly graded filtered module over R (with the module structure consistent with . . .), then we may form the doubly graded module $E_0(A)$ over $E_0(R)$.

For the present purposes it suffices to consider filtrations such that

$$F^iR^n = 0 \qquad \text{for } i > n.$$

Since this also insures that the resulting double grading on $E_0(R)$ is non-negative we shall establish it as a convention.

PROPOSITION 2.1. *Let R be a filtered ring and A a filtered module (with the above conventions) over R. If $E_0(A)$ is (left) noetherian over $E_0(R)$, then A is noetherian over R.*

Proof. See [6, Chapter VIII, §3, Corollary 3, p. 260]. This proof is made in somewhat different circumstances but remains valid in our case.

REMARK. When dealing with graded rings, in order that a graded module be noetherian, it is sufficient that every ascending chain of *graded* submodules terminate. This fact is related to Proposition 2.1 and in fact is a consequence of a modification of this proposition.

Let R be a ring and A a module over it. Define

$$R[X] = Z[X] \otimes R,$$
$$A[X] = Z[X] \otimes A.$$

Notice that $A[X]$ becomes a module over the ring $R[X]$ through the operation

$$(f(X) \otimes r)(g(X) \otimes a) = f(X)g(X) \otimes ra, \qquad r \in R,\ a \in A.$$

PROPOSITION 2.2. (HILBERT BASIS THEOREM). *If A is noetherian over R then $A[X]$ is noetherian over $R[X]$.*

Proof. As usual.

COROLLARY 2.3. *Let R be a ring and A a module over it. If R contains a subring R' and A contains a noetherian R'-submodule A' such that*

$$A = \sum_{n \geq 0} x^n A' \quad (\textit{not necessarily direct})$$

for some x in the center of R, then A is R-noetherian.

For graded rings the following converse of the Hilbert Basis Theorem is true.

PROPOSITION 2.4. *Let $R = \sum_{n \geq 0} R^n$ be a graded ring (with our conventions). If R is noetherian, then R^0 is noetherian, and R is finitely generated as a ring over R^0 by homogeneous elements.*

Proof. The first contention is clear since R^0 is a homomorphic image of R. Let A be the ideal of elements of positive degree. Writing

$$A = Ra_1 + Ra_2 + \cdots + Ra_n$$

for some choice of homogeneous elements $a_1, a_2, \cdots, a_n$ and using

$$R = R^0 \oplus A,$$

we see that R is generated as a ring over R^0 by $1, a_1, a_2, \cdots, a_n$. q.e.d.

3. **The spectral sequences of Hochschild-Serre.** Let G be a group and k a ring on which G acts as a group of ring automorphisms. (Say that k is a (G)-ring.) Let H be a normal subgroup of G. Then according to [4] there is a spectral sequence

$$H^*(G/H, H^*(H, k)) \cong E_2(k), E_3(k), \cdots, E_r(k), \cdots,$$
$$E_\infty(k) \cong E_0(H^*(G, k)),$$

where $H^*(G, k)$ is suitably filtered. Moreover, each $E_r(k)$ is a (doubly graded) ring, each d_r satisfies the product rule, and $E_{r+1}(k)$ is the homology ring of $E_r(k)$. Also, since each $E_\infty^{i,j}(k) = E_r^{i,j}(k)$ for r sufficiently large, $E_\infty(k)$ inherits a product structure and is isomorphic to $E_0(H^*(G, k))$ as a ring.

Since $H^*(H, k)$ is a singly graded (G/H)-ring, $H^*(G/H, H^*(H, k)) = \sum_{p,q \geq 0} H^p(G/H, H^q(H, k))$ has a doubly graded ring structure. By [4] the above isomorphism with $E_2(k)$ is an isomorphism of rings *except for a sign.* This will make no essential difference in what follows. However since $H^*(H, k)$

is a graded ring, the ordinary cup product in $H^*(G/H, H^*(H, k))$ ought to be redefined to include a sign convention. Supposing this to be done, we may assume that this isomorphism is a ring isomorphism.

Finally, with respect to edge homomorphisms we have commutative diagrams

$$\text{(I)}\qquad \begin{array}{ccc} H^*(G/H, k^H) & \xrightarrow{\text{inflation}} & H^*(G, k) \\ \cong\downarrow & & \uparrow \quad \text{(injection)} \\ E_2^{*,0} & \xrightarrow{\text{(surjection)}} & E_\infty^{*,0} \end{array}$$

$$\text{(II)}\qquad \begin{array}{ccc} H^*(G, k) & \xrightarrow{\text{restriction}} & H^*(H, k)^G \subseteq H^*(H, k) \\ \text{(surjection)}\downarrow & & \uparrow\cong \\ E_\infty^{0,*} & \xrightarrow[\text{(injection)}]{} & E_2^{0,*}. \end{array}$$

Suppose now that M is a module over k on which G acts in such a way that the natural pairing

$$k \otimes M \rightarrow M$$

is a G-pairing. (Say that M is a (G)-module over k.) Then also by [4] there exists a spectral sequence

$$H^*(G/H, H^*(H, M)) \cong E_2(M), E_3(M), \cdots, E_r(M), \cdots,$$
$$E_\infty(M) \cong E_0(H^*(G, M)),$$

where each $E_r(M)$ is a module over $E_r(k)$ and considerations such as above apply. Also $H^*(G/H, H^*(H, M))$ and $H^*(G, M)$ are modules over the appropriate rings (in fact, with some additional structure such as grading or filtration), and the isomorphisms of the second spectral sequence are consistent with the ring isomorphisms of the first. (Again a sign convention is needed for E_2.)

In the future for convenience we will treat the spectral sequence isomorphisms as identifications.

4. **Some technical preliminaries.** Let H be a cyclic group of finite order h and let σ be a generator. Let k be an (H)-ring and M an (H)-module over it. According to [2, Chapter XII, §7] we have isomorphisms

$$H^0(H, k) \cong k^H,$$
$$H^{2n}(H, k) \cong k^H/N_H k, \qquad n > 0,$$
$$H^{2n+1}(H, k) \cong k_{N_H}/(\sigma - 1)k, \qquad n \geqq 0.$$

Moreover, for specified σ, these are natural isomorphisms.

Furthermore, if $c \in k$ determines $\gamma \in H^i(H, k)$ and $m \in k$ determines $\mu \in H^j(H, k)$, then $\gamma\mu \in H^{i+j}(H, k)$ is determined by

$$\begin{cases} cm, & i \text{ even or } j \text{ even}, \\ \sum_{0 \leq q < r < h} (\sigma^q c)(\sigma^r m), & i \text{ and } j \text{ odd}. \end{cases}$$

Corresponding statements apply to M (choosing $m \in M$, etc.).

Let $\chi_k \in H^2(H, k)$ be determined by $1 \in k^H$. This element will depend of course on the above isomorphism and thus ultimately on the choice of generator of H. Since we shall be interested only in what happens when we change the ring k, we shall suppress this dependence (assuming that some fixed choice has been made at the beginning of the discussion).

It is clear that multiplication by χ_k gives rise to an isomorphism

$$H^q(H, M) \to H^{q+2}(H, M) \qquad q > 0.$$

As a result we have

PROPOSITION 4.1. *Let H be a finite cyclic group, k an (H)-ring, and M an (H)-module over k. $H^*(H, M)$ is noetherian over $H^*(H, k)$ if and only if M^H and $M_{N_H}/(\sigma-1)M$ are noetherian over k^H.*

Proof. Since

$$H^*(H, M) = \sum_{n \geq 0} \chi_k^n [H^0(H, M) + H^1(H, M)],$$

one implication follows from Corollary 2.3. Conversely, $H^0(H, M) + H^1(H, M)$ is a homomorphic image of $H^*(H, M)$. q.e.d.

Let G be a finite group, H a normal cyclic subgroup, k a (G)-ring, and M a (G)-module over k. As in §3 we have spectral sequences. The following special fact will be useful in the sequel.

PROPOSITION 4.2. (*Let χ_k be as above.*) *If $\chi_k \in H^2(H, k)^G$ then multiplication by χ_k gives rise to an isomorphism*

$$E_2^{*,j}(M) \to E_2^{*,j+2}(M), \qquad j > 0,$$

of singly graded abelian groups.

Proof. If $\chi_k \in H^2(H, k)^G$, then the isomorphism

$$H^j(H, M) \xrightarrow{\chi_k} H^{j+2}(H, M), \qquad j > 0,$$

gives rise to isomorphisms

$$H^i(G/H, H^j(H, M)) \to H^i(G/H, H^{j+2}(H, M)), \qquad i \geqq 0, j > 0.$$

For $i=0$ this induced map is multiplication by χ_k, and one sees by use of explicit cochain formulas that this is true for $i>0$. q.e.d.

Let $\eta: Z \rightarrow k$ be defined by $\eta(n)=n\cdot 1$. It induces a homomorphism

$$\eta^*: H^*(G/H, H^*(H, Z)) = E_2(Z) \rightarrow E_2(k).$$

(We shall suppose that the ring of integers Z is given always with trivial action of the group under consideration.) η^* will be used to reduce some questions about arbitrary rings k to questions about Z. In particular we need the following fact.

PROPOSITION 4.3. *If G operates trivially on $H^2(H, Z)$, then $\eta^*(\chi_Z)=\chi_k$ is in $H^2(H, k)^G$ and also in the center of the ring $E_2(k)$.*

Proof. Since the operation of G commutes with η^*, the first contention is clear. The second follows from the formula

$$\eta^*(x)\cdot y = (-1)^{(i+j)(i'+j')} y\cdot\eta^*(x),$$

where $x\in E_2^{i,j}(Z)$ and $y\in E_2^{i',j'}(k)$. This formula is true since $\eta(Z)$ is contained in the center of k and with the appropriate sign convention for products in $H^*(G/H, H^*(H, k))$ an anticommutativity rule holds. q.e.d.

5. **The main lemma.** Let G be a finite group and H a cyclic subgroup of order h contained in the center of G. Then G acts trivially on $H^2(H, Z)$, and we may consider the situation discussed in the previous section. We have

$$d_2\chi_Z^h = 0, \qquad d_3\chi_Z^{h^2} = 0, \text{ etc.}$$

so that it is reasonable to ask whether $\chi_Z^l\in E_\infty^{0,2l}$ for some integer $l>0$. This is in fact so, and as a result we can prove that $E_s=E_\infty$ for some $s<\infty$. It becomes therefore much easier to study $H^*(G, Z)$ via the spectral sequence.

We shall now prove this fact by imbedding G in a certain "wreath product," thereby reducing the problem to a question about semi-direct products.

Suppose now that G is a finite group and H is any subgroup of its center. Denote by S_G the group of permutations of the elements of G. Letting G act on itself by left translation, we may view G as imbedded in S_G.

Let R be the centralizer of H in S_G. (Clearly $G\subset R$.) The elements ρ of R are characterized by the property

$$\rho(\sigma\tau) = \sigma\rho(\tau), \qquad \sigma\in H, \tau\in G. \tag{1}$$

Thus R consists of those elements permuting the cosets of H in G among themselves.

Let $\tau_1, \tau_2, \cdots, \tau_l$ be representatives, one from each of the cosets of H. By virtue of (1) any element of R is determined by what it does to the τ_i. In particular, let S_l be the subgroup of R which just permutes the τ_i among themselves. Clearly S_l may be identified with the symmetric group on l letters; that is we may write $\rho(\tau_i)=\tau_{\rho(i)}$.

Let H_i be the subgroup of R whose action on the set $\tau_i H$ coincides with that of H and whose action elsewhere is the identity. Since the H_i are essentially groups of permutations of disjoint sets it is clear that

$$\overline{H} = H_1 H_2 \cdots H_l.$$

is a *direct* product. Since each H_i is naturally isomorphic with H it will lead to no confusion to consider $\overline{H}$ as the external direct product

$$H \times H \times \cdots \times H$$

of l copies of H.

Given $\rho \in R$ we have

$$\rho(\tau_i) = \sigma_i \rho'(\tau_i) \quad \text{where} \quad \rho' \in S_l,\ \sigma_i \in H_i,\ i = 1, 2, \cdots, l,$$

and ρ is completely determined by ρ' and the σ_i. Thus ρ can be written uniquely

$$\rho = \rho' \sigma_1 \sigma_2 \cdots \sigma_l.$$

Moreover, using the above convention for $\overline{H}$, we can describe the operation of S_l on $\overline{H}$ by

$$(2) \qquad \rho^{-1}(\sigma_1 \times \sigma_2 \times \cdots \times \sigma_l)\rho = (\sigma_{\rho(1)} \times \sigma_{\rho(2)} \times \cdots \times \sigma_{\rho(l)})$$

where $\sigma_1, \sigma_2, \cdots, \sigma_l \in H$ and $\rho \in S_l$. Thus the centralizer R of H is a semidirect product of S_l with the normal subgroup $\overline{H}$.

We are now in a position to prove

PROPOSITION 5.1. *Let G be a finite group and H a subgroup in the center of G. If $\chi \in H^*(H, Z)$ is homogeneous of even degree, then*

$$\chi^{(G:H)} \in \operatorname{Im}\{\text{res}: H^*(G, Z) \to H^*(H, Z)\}.$$

Proof. Because of the above construction and because of the transitivity of the restriction homomorphism it suffices to prove the proposition for the case where G is the group R above and H is its center.

We shall construct a special R-projective resolution of Z with which to compute.

Let X be any H-projective resolution of Z. Then (as in [2, Chapter IX, 2.7]) the complex

$$\overline{X} = X \otimes X \otimes \cdots \otimes X \quad (l \text{ times})$$

is an $\overline{H} = H \times H \times \cdots \times H$-projective resolution of Z. Here $\overline{H}$ operates by

$$(3) \quad (\sigma_1 \times \sigma_2 \times \cdots \times \sigma_l)(x_1 \otimes x_2 \otimes \cdots \otimes x_l) = \sigma_1 x_1 \otimes \sigma_2 x_2 \otimes \cdots \otimes \sigma_l x_l.$$

Given $\rho\in S_l$, $x_i\in X_{p_i}$ $(i=1, 2, \cdots, l)$, define

(4) $\rho(x_1\otimes x_2\otimes\cdots\otimes x_l) = (-1)^{\nu(\rho;p_1,p_2,\cdots,p_l)}(x_{\rho^{-1}(1)}\otimes x_{\rho^{-1}(2)}\otimes\cdots\otimes x_{\rho^{-1}(l)})$

where $\nu(\rho; p_1, p_2, \cdots, p_l)$ is defined mod 2 as follows: set

$$T(i,j) = \begin{cases} 1, & \text{if } i<j, \\ -1, & \text{if } i>j, \end{cases}$$

and let

$$(-1)^{\nu(\rho;p_1,p_2,\cdots,p_l)} = \prod_{1\leq i<j\leq l} T(\rho i, \rho j)^{p_ip_j}.$$

(That is, except for sign ρ naturally permutes the factors, and the sign is chosen so that ρ commutes with the differentiation in $\overline{X}$.)

The following facts may be checked by explicit computation. Notice that it suffices to check (ii) for transpositions of the form $(1j)$.

(i) The definition (4) gives rise to an operation of S_l on $\overline{X}$ as a group of automorphisms.

(ii) The operation of S_l commutes with the differentiation in $\overline{X}$ and with the natural augmentation of this complex as a complex over Z.

(iii) The operations of S_l and $\overline{H}$ on $\overline{X}$ are consistently related so that $\overline{X}$ may be viewed in a natural way as a left R-complex over Z.

Let Y be an S_l-projective resolution of Z. Letting

$$(\bar{\sigma}\rho)(\bar{x}\otimes y) = \bar{\sigma}\rho\bar{x}\otimes\rho y, \qquad \bar{\sigma}\in\overline{H}, \rho\in S_l, \bar{x}\in\overline{X}, y\in Y,$$

we view

$$W = \overline{X}\otimes Y$$

as an R-complex over Z.

W is in fact an R-projective resolution of Z. Since it is obviously acyclic, in order to demonstrate this fact it suffices to show that each $\overline{X}_q\otimes Y_n$ is R-projective. Suppose that we have

$$\begin{array}{c} \overline{X}_q\otimes Z(S_l) \\ \downarrow f \\ A' \xrightarrow{j} A \to 0 \end{array}$$

with f, j R-homomorphisms and j surjective. Since $\overline{X}_q$ is H-projective, there exists an H-homomorphism $g: \overline{X}_q\otimes 1\to A'$ such that $jg=f$. Define an R-homomorphism $g': \overline{X}_q\otimes Z(S_l)\to A'$ by

$$g'(x\otimes\tau) = \tau g(\tau^{-1}x\otimes 1).$$

It is easy to verify that $jg'=f$. Thus $\overline{X}_q\otimes Y_r$ is R-projective for Y_r S_l-free. The more general result follows by a direct sum argument.

Now suppose that $\chi \in H^{2n}(H, Z)$ and that $f \in \mathrm{Hom}_H(X_{2n}, Z)$ is a cocycle representing χ. Let ϵ: $Y \to Z$ be the augmentation for the resolution Y. (Remember that ϵ represents the unit element of the ring $H^*(S_l, Z)$.) Then

$$f \otimes f \otimes \cdots \otimes f \otimes \epsilon \in \mathrm{Hom}_R(W_{2nl}, Z)$$

is a cocycle representing some $\zeta \in H^{2nl}(R, Z)$. The element

$$\xi = \mathrm{res}_{R \to \overline{H}}(\zeta) \in H^{2nl}(\overline{H}, Z)$$

is represented by the cocycle

$$f \otimes f \otimes \cdots \otimes f \in \mathrm{Hom}_{\overline{H}}(\overline{X}_{2nl}, Z).$$

Under the identification

$$\overline{H} = H \times H \times \cdots \times H$$

the natural injection of H in $\overline{H}$ becomes the diagonal imbedding

$$\Delta: H \to H \times H \times \cdots \times H$$

defined by $\Delta(\sigma) = \sigma \times \sigma \times \cdots \times \sigma$. In cohomology, the restriction from $\overline{H}$ to H becomes the map induced by Δ.

On the other hand we may split up the l-fold cup product, $\chi \to \chi^l$, by means of the diagram

$$\begin{array}{ccc} H^*(H, Z) \otimes H^*(H, Z) \otimes \cdots \otimes H^*(H, Z) & \longrightarrow & H^*(H, Z) \\ \downarrow & \nearrow_{\Delta^*} & \\ H^*(H \times H \times \cdots \times H, Z) & & \end{array}$$

where the vertical map is the so-called *external* product. Since on the cochain level $f \otimes f \otimes f \otimes \cdots \otimes f$ defines the l-fold external product of χ with itself it follows that

$$\mathrm{res}_{\overline{H} \to H}(\xi) = \Delta^*(\xi) = \chi^l.$$

Putting together the two statements, we complete the proof. q.e.d.

REMARK. For a construction analogous to the construction of W see [7].

6. The main theorem.

THEOREM 6.1. *Let G be a finite group, k a ring on which G acts trivially, and M a (G)-module over k. If M is noetherian over k, then $H^*(G, M)$ is noetherian over $H^*(G, k)$.*

Proof. We divide the proof into two parts.

Case (1). G is nilpotent.

We proceed by induction on $g = (G: 1)$. The theorem is a tautology for

$g=1$. Suppose it is true for all nilpotent G' with $(G'{:}\ 1)<g$, and let G be nilpotent with $1<(G{:}\ 1)$. Let k and M be given satisfying the hypotheses for G.

Since the center of G is nontrivial, it contains a nontrivial cyclic subgroup H. Let $l=(G{:}\ H)$. With the notation as in §4 we can deduce from 5.1 that

$$\overset{l}{\chi_Z} \in \operatorname{Im} \operatorname{res}_{G\to H}.$$

so that

$$\eta^*(\overset{l}{\chi_Z}) = \overset{l}{\chi_k} \in \operatorname{Im} \operatorname{res}_{G\to H}.$$

Write $\xi=\chi_k^l \in H^{2l}(H, k)$.

By [4], we have a pair of spectral sequences

$$H^*(G/H, H^*(H, k)) = E_2(k), E_3(k), \cdots, E_\infty(k) \cong E_0(H^*(G, k))$$

and

$$H^*(G/H, H^*(H, M)) = E_2(M), E_3(M), \cdots, E_\infty(M) \cong E_0(H^*(G, M)).$$

Also the ring and module structures are as discussed in §3. Notice that we may make the identification

$$\operatorname{Im} \operatorname{res}_{G\to H} = E_\infty^{0,*}.$$

Since G acts trivially on $H^2(H, Z)$ we may apply 4.2, and thus also 4.3, so that ξ is in the center of $E_2(k)$ and

$$E_2(M) = \sum_{n\geq 0} \xi^n E_2'(M) \qquad \text{(not direct)}$$

where

$$E_2'(M) = \sum_{0\leq q\leq 2l} E_2^{*,q}(M).$$

Also $E_2'(M)$ is noetherian over $E_2' = E_2^{*,0}(k) = H^*(G/H,k)$. For, each $H^q(H, M)$ is certainly noetherian over k, and, by the induction hypothesis applied to G/H, each

$$E_2^{*,q}(M) = H^*(G/H, H^q(H, M))$$

is noetherian over $H^*(G/H, k)$. We have checked the hypotheses of 2.3 and may conclude that $E_2(M)$ is noetherian over $E_2'[\xi]$. Let

$$Z_{\infty,2}(M) = \{x \in E_2(M) \mid d_2x = d_3x = \cdots = 0\}.$$

Since $d_2\xi=d_3\xi= \cdots =0$ also, by the product rule we see that $Z_{\infty,2}(M)$ is an $E_2'[\xi]$-submodule of $E_2(M)$. It is therefore noetherian over $E_2'[\xi]$.

Write $E_\infty' = E_\infty^{*,0}(k)$; we have a ring homomorphism

$$E_2'[\xi] \to E_\infty'[\xi]$$

and a homomorphism of modules consistent with it

$$E_{\infty,2}(M) \to E_\infty(M).$$

Since these are surjections it follows that $E_\infty(M)$ is noetherian over $E_\infty'[\xi]$ and thereby over $E_\infty(k)$. Finally, by 2.1 it follows that $H^*(G, M)$ is noetherian over $H^*(G, k)$.

REMARK. Notice that the above also implies that $E_r(M) = E_\infty(M)$ for some $r < \infty$. I feel that this is the essential fact under consideration but that it is masked by the arrangement of the proof.

Case (2). G is an arbitrary finite group.

Let $g = (G: 1)$ and for each prime p let g_p be the exact power of p dividing g.

Define

$$\overline{H}(G, k; p) = H^*(G, k)/g_p H^*(G, k)$$

and similarly for M define the corresponding module over this ring. Then modulo g each $\overline{H}(G, M; p)$ can be viewed as a module over $H^*(G, k)$; also modulo g, $H^*(G, M)$ is generated by the $\overline{H}(G, M; p)$. Since

$$gH^*(G, M) = gH^0(G, M) = gM^G,$$

in order to show that $H^*(G, M)$ is noetherian over $H^*(G, k)$, it is sufficient to show that $\overline{H}(G, M; p)$ is noetherian over $\overline{H}(G, k; p)$ for each prime p dividing g.

Fix such a p and let G_p be a p-Sylow subgroup of G. We may suitably modify the ordinary restriction and transfer homomorphisms so as to define in a natural way homomorphisms

$$\text{res}: \overline{H}(G, k; p) \to \overline{H}(G_p, k) = H^*(G_p, k)/g_p H^*(G_p, k)$$
$$\text{tr}: \overline{H}(G_p, k) \to \overline{H}(G, k; p)$$

and similarly for M. One checks easily that the formulas (6), (11), and (12) of [2, Chapter XII, §8, pp. 285–286] hold for the modified maps, i.e.

(a) $\quad \text{tr} \circ \text{res} = g/g_p,$

(b) $\quad \text{res}\,(r\,a) = \text{res}(r)\cdot\text{res}(a),$

$\qquad r \in \overline{H}(G, k; p)$ and $a \in \overline{H}(G, k; p)$ or $\overline{H}(G, M; p)$,

(c) $\quad \text{tr}(r'\cdot \text{res}\, a) = (\text{tr}\, r')\cdot a, \qquad r' \in \overline{H}(G_p, k; p)$ and $a \in \overline{H}(G, M; p)$.

Write

$$R(k) = \text{Im}\{\text{res}: \overline{H}(G, k; p) \to \overline{H}(G_p, k)\},$$
$$T(k) = \text{Ker}\{\text{tr}: \overline{H}(G_p, k) \to \overline{H}(G, k; p)\},$$

and similarly for M. As in [2, Chapter XII, 10.1], it follows from (a) and (b) that restriction yields a ring isomorphism

$$\bar{H}(G, k; p) \cong R(k) \tag{1}$$

and a corresponding module isomorphism

$$\bar{H}(G, M; p) \cong R(M). \tag{1'}$$

Moreover, we have decompositions

$$\begin{aligned} \bar{H}(G_p,\ k) &= R(k) \oplus T(k), \\ \bar{H}(G_p, M) &= R(M) \oplus T(M) \end{aligned}$$

(of abelian groups). From (c) it follows that

$$T(k)\cdot R(M) \subseteq T(M). \tag{2}$$

By (1) and (1′) it is sufficient to show that $R(M)$ is noetherian over $R(k)$. Let

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots \tag{3}$$

be an ascending chain of $R(k)$-submodules of $R(M)$. Set

$$B_n = \bar{H}(G_p, k)\cdot A_n.$$

Then

$$B_1 \subseteq B_2 \subseteq \cdots \subseteq B_n \subseteq \cdots \tag{4}$$

is an ascending chain of $\bar{H}(G_p,\ k)$-submodules of $\bar{H}(G_p,\ M)$. By *Case* (1), $H^*(G_p,\ M)$ is noetherian over $H^*(G_p,\ k)$. This certainly remains true for our modified module and ring so that the chain (4) contains only a finite number of distinct terms. That this is true also of the chain (3) follows from the following lemma.

LEMMA. *Let $A' \subseteq A$ be $R(k)$-submodules of $R(M)$. If $\bar{H}(G_p, k)A' = \bar{H}(G_p, k)A$, then $A' = A$.*

Proof.

$$\begin{aligned} B = \bar{H}(G_p, k)A &= \{R(k) \oplus T(k)\}A \\ &= R(k)A + T(k)A, \end{aligned}$$

where the second sum could not be direct (in principle). However, by (b) and (2)

$$\begin{aligned} R(k)A &\subseteq R(M), \\ T(k)A &\subseteq T(M), \end{aligned}$$

so that in fact

$$B = R(k)A \oplus T(k)A.$$

Similarly

$$B' = \overline{H}(G_p, k)A' = R(k)A' \oplus T(k)A'.$$

Since

$$R(k)A' \subseteq R(k)A,$$
$$T(k)A' \subseteq T(k)A,$$

the lemma and hence the theorem follow. q.e.d.

COROLLARY 6.2. *Let G be a finite group and k a ring on which G acts trivially. If k is noetherian then $H^*(G, k)$ is a finitely generated ring over k.*

7. **Some consequences of the main theorem.** Let G be a finite group and H a subgroup. If k is a ring on which G acts trivially and N is an (H)-module over k, then $H^*(H, N)$ may be viewed as a module over $H^*(G, k)$ by means of the ring homomorphism

$$\text{res}: H^*(G, k) \to H^*(H, k).$$

The following extension of the main theorem is true.

THEOREM 7.1. *Let G be a finite group, H a subgroup, k a ring on which G acts trivially, and N an (H)-module over k. If N is noetherian over k, then $H^*(H, N)$ is noetherian over $H^*(G, k)$.*

Proof. Consider the induced (G)-module over k

$$M = \text{Hom}_H(Z(G), N),$$

where

$$(\sigma m)(\tau) = m(\tau\sigma), \qquad m \in M, \sigma, \tau \in G,$$
$$(cm)(\tau) = c(m(\tau)), \qquad c \in k, m \in M, \tau \in G.$$

Let

$$\pi: M \to N$$

be defined by $\pi(m) = m(1)$, $m \in M$. Notice that π is a homomorphism of (H)-modules over k. Then

$$\phi = \pi^* \circ \text{res}_{G \to H}: H^*(G, M) \to H^*(H, N)$$

is an isomorphism. (See [2, Chapter X, 7.4] for an analogous statement.)

Since π is a k-homomorphism, π^* is an $H^*(H, k)$-homomorphism. Thus for $r \in H^*(G, k)$ and $a \in H^*(G, M)$

$$\begin{aligned}\phi(ra) &= \pi^*(\text{res } ra) = \pi^*(\text{res } r\cdot\text{res } a)\\ &= \text{res } r\cdot\pi^*(\text{res } a)\\ &= \text{res } r\cdot\phi(a).\end{aligned}$$

In other words ϕ is an $H^*(G, k)$-homomorphism.

Since $M=\text{Hom}_Z(Z(G), N)^H$ it follows that M is k-noetherian if N is. Then $H^*(G, M)$ is noetherian over $H^*(G, k)$ by the main theorem, and this is also true for $H^*(H, N)$. q.e.d.

COROLLARY 7.2. *Let G be a finite group and H a cyclic subgroup of prime order. With the notation of* §4,

$$\chi_Z^l \in \text{Im}\{\text{res}: H^{2l}(G, Z) \to H^{2l}(H, Z)\}$$

for some integer $l>0$. In particular, for any nontrivial finite group there is an integer $l>0$ such that

$$H^{2q}(G, Z; p) \neq 0 \qquad q \equiv 0 \mod l.$$

Proof. By Theorem 7.1, $H^*(H, Z)$ is a finite module over the subring Im $\text{res}_{G\to H}$. Since $H^*(H, Z)$ is essentially the ring of polynomials in χ_Z with coefficients in $Z/(H:1)Z$ it is certainly not a finite module over $H^0(H, Z)$. q.e.d.

COROLLARY 7.3. *Let G be a finite group, H a subgroup of prime order, and L a G-module. If $L^G \not\subseteq (\sum_{\sigma\in H}\sigma)L$, then there is an integer $l>0$ such that*

$$\text{res}: H^{2q}(G, L) \to H^{2q}(H, L) \qquad q \equiv 0 \mod l$$

is nontrivial. In particular, if G acts trivially on L then the conclusion is valid for arbitrary subgroups H such that $(H:1)L\neq L$.

Proof. The hypothesis says that the homomorphism

$$\text{res}: \hat{H}^0(G, L) \to \hat{H}^0(H, L) \tag{1}$$

is nonzero where $\hat{H}^0$ is an ordinary 0-degree group modulo norms. If α is in the image of (1) and not zero, then

$$\chi_Z^q\alpha \in \text{Im res}_{G\to H} \qquad q \equiv 0 \mod l$$

(where l is as in 7.2), and this element is certainly not zero. q.e.d.

REMARK. That this generalization of the main lemma is true was pointed out to me by R. G. Swan. For a topological proof see reference [8] which deals also with compact Lie groups. That for a finite group the stronger property stated in 7.1 is true occurred to me later. Also the present proof is completely algebraic—which answers a question of Swan.

It can be proved easily by methods outside the scope of this paper that the cohomology ring of a group characterizes the group in the following *weak* sense. If $\phi: G \to G'$ is a group homomorphism and $\phi^*: H^*(G', Z) \to H^*(G, Z)$ is an isomorphism, then ϕ is an isomorphism. About the question of stronger characterization we can make the following limited statement as a corollary to 7.3.

COROLLARY 7.4. *Let G be a finite group. G is abelian if and only if $H^*(G, Z)$ is generated as a ring by elements of degree $\leqq 3$.*

Proof. One implication is known. For the converse let H be a cyclic subgroup contained in the derived subgroup G'. The restriction in dimension 2 is dual to the natural map (induced by inclusion)

$$H/H' = H \to G/G'$$

which is trivial. Thus under our hypothesis the restriction is trivial in all positive dimensions. This contradicts 7.3 unless H is trivial. Since H was an arbitrary cyclic subgroup of G' to start with, G' must be trivial. q.e.d.

8. **The commutative case.** If k is a commutative (G)-ring, then $H^*(G, k)$ is an anti-commutative graded ring and certain simplifications occur. The distinction between right and left modules virtually disappears. Also, if $H^*(G, k)$ is noetherian then it is a homomorphic image of the free anti-commutative ring over k^G in a certain finite number of generators.

In the general case, a simple counterexample can be constructed to show that the main theorem cannot be extended to nontrivial action on k. In the commutative case, however, it is possible in some cases to relax this restriction.

THEOREM 8.1. *Let G be a finite group, k a commutative (G)-ring, and M a (G)-module over k. Suppose that k is finitely generated as a ring over a noetherian subring k' contained in k^G. Then $H^*(G, k)$ is finitely generated as a ring over k'; also if M is a finitely generated module over k, then $H^*(G, M)$ is a finitely generated module over $H^*(G, k)$.*

Proof. Let $k = k'[x_1, x_2, \cdots, x_n]$; then each x_i satisfies a polynomial

$$f_i(X) = \prod_{\sigma \in G} (X - \sigma x_i)$$

with coefficients in k^G. It follows that k is a finitely generated module over k^G. By a theorem of Artin and Tate [1], we may conclude that k^G is a finitely generated ring over k'. Since under the above hypotheses k and M are finitely generated modules over k^G, the main theorem implies that the corresponding cohomology modules are finitely generated over $H^*(G, k^G)$ and hence also over $H^*(G, k)$. q.e.d.

Let G, k, and M be as above. We can consider the "Poincaré function"

$$p_G(M; X) = \sum_{q=0}^{\infty} b_q X^q, \tag{1}$$

where

$$b_q = d_k(H^q(G, M)),$$

d_k being some "dimension" function. Kostrikin and Šafarevič [5] have conjectured that the function corresponding to (1) for the case of "nilpotent algebras" is a rational function when M is $Z/p\,Z$. Golod and Venkov in the previously mentioned papers remark that finite generation of the cohomology ring implies the truth of the conjecture for the various classes of groups which they consider. For finite groups, the results contained in this paper imply the corresponding statement for finitely generated coefficient modules.

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